

VALUE REGIONS OF UNIVALENT SELF-MAPS WITH TWO BOUNDARY FIXED POINTS

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ABSTRACT. In this paper we find the exact value region $\mathcal{V}(z_0, T)$ of the point evaluation functional $f \mapsto f(z_0)$ over the class of all holomorphic injective self-maps $f : \mathbb{D} \rightarrow \mathbb{D}$ of the unit disk \mathbb{D} having a boundary regular fixed point at $\sigma = -1$ with $f'(-1) = e^T$ and the Denjoy–Wolff point at $\tau = 1$.

1. INTRODUCTION

Since the seminal paper [11] by Cowen and Pommerenke, the study of holomorphic functions with finite angular derivative at prescribed boundary points has been an active field of research in Complex Analysis, see, e.g., [2, 3, 10, 15, 17, 31, 36], just to mention some works in the topic.

Given a holomorphic function f in the unit disk $\mathbb{D} := \{z : |z| < 1\}$ and a point $\sigma \in \partial\mathbb{D}$ such that there exists finite angular limit $f(\sigma) := \angle \lim_{z \rightarrow \sigma} f(z)$, the *angular derivative* at σ is $f'(\sigma) := \angle \lim_{z \rightarrow \sigma} (f(z) - f(\sigma))/(z - \sigma)$.

On the one hand, for univalent (i.e., holomorphic and injective) functions f , existence of the angular derivative $f'(\sigma)$ different from 0 and ∞ is closely related to the geometry of $f(\mathbb{D})$ near $f(\sigma)$; moreover, if there exists $f'(\sigma) \neq 0, \infty$, then the behaviour of f at the boundary point σ resembles conformality, see, e.g., [30, §§4.3, 11.4].

On the other hand, for the dynamics of a holomorphic (but not necessarily univalent) self-map $f : \mathbb{D} \rightarrow \mathbb{D}$, a crucial role is played by the points $\sigma \in \partial\mathbb{D}$ for which $f(\sigma) = \sigma$ (or, more generally, $f(\sigma) \in \partial\mathbb{D}$) and the angular derivative $f'(\sigma)$ is finite, see, e.g., [5–7, 8, 9, 14, 16, 29]. Such points σ are called *boundary regular fixed points*, see Section 2 for precise definitions and some basic theory. In particular, a classical result due to Wolff and Denjoy asserts that if $f \in \text{Hol}(\mathbb{D}, \mathbb{D})$ has no fixed points in \mathbb{D} , then it possesses the so-called (*boundary*) *Denjoy–Wolff point*, i.e., a unique boundary regular fixed point τ such that $f'(\tau) \leq 1$.

In this paper we study *univalent* self-maps $f : \mathbb{D} \rightarrow \mathbb{D}$ with a given boundary regular fixed point $\sigma \in \partial\mathbb{D}$ and the Denjoy–Wolff point $\tau \in \partial\mathbb{D} \setminus \{\sigma\}$. Using automorphisms of \mathbb{D} , we may suppose that $\tau = 1$ and $\sigma = -1$. Our main result is the *sharp value region* of $f \mapsto f(z_0)$ for all such self-maps of \mathbb{D} with $f'(-1)$ fixed. To give a detailed statement, fix $z_0 \in \mathbb{D}$, $T > 0$ and let $\zeta_0 = x_1^0 + ix_2^0 := \ell(z_0)$, where

$$\ell : \mathbb{D} \rightarrow \mathbb{S}; \quad z \mapsto \log((1+z)/(1-z))$$

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is a conformal map of \mathbb{D} onto the strip $\mathbb{S} := \{\zeta : -\pi/2 < \operatorname{Im} \zeta < \pi/2\}$. Define:

$$a_{\pm}(T) := e^{-T/2} \sin x_2^0 \pm (1 - e^{-T/2}), \quad R(a, T) := \log \frac{1-a}{1-a_+(T)} \log \frac{1+a}{1+a_-(T)},$$

$$V(\zeta_0, T) := \left\{ x_1 + ix_2 \in \mathbb{S} : a_-(T) \leq \sin x_2 \leq a_+(T), \quad |x_1 - x_1^0 - \frac{T}{2}| \leq \sqrt{R(\sin x_2, T)} \right\}.$$

Theorem 1. *Let $f \in \operatorname{Hol}(\mathbb{D}, \mathbb{D}) \setminus \{\operatorname{id}_{\mathbb{D}}\}$ and $T > 0$. Suppose that:*

- (i) *f is univalent in \mathbb{D} ;*
- (ii) *the Denjoy – Wolff point of f is $\tau = 1$;*
- (iii) *$\sigma = -1$ is a boundary regular fixed point of f and $f'(-1) = e^T$.*

Then

$$(1.1) \quad f(z_0) \in \mathcal{V}(z_0, T) := \ell^{-1}(V(\ell(z_0), T)) \setminus \{z_0\} \quad \text{for any } z_0 \in \mathbb{D}.$$

This result is sharp, i.e., for any $w_0 \in \mathcal{V}(z_0, T)$ there exists $f \in \operatorname{Hol}(\mathbb{D}, \mathbb{D}) \setminus \{\operatorname{id}_{\mathbb{D}}\}$ satisfying (i) – (iii) and such that $f(z_0) = w_0$.

We can also characterize functions f delivering boundary points of $\mathcal{V}(z_0, T)$. In many extremal problems for univalent functions $f : \mathbb{D} \rightarrow \mathbb{C}$ normalized by $f(0) = f'(1) - 1 = 0$, the Koebe function $f_0(z) := z/(1-z)^2$ mapping \mathbb{D} onto $\mathbb{C} \setminus (-\infty, \frac{1}{4}]$, and its rotations $f_{\theta}(z) = e^{i\theta} f_0(e^{-i\theta} z)$, $\theta \in \mathbb{R}$, are known to be extremal. For bounded univalent functions $f : \mathbb{D} \rightarrow \mathbb{D}$ normalized by $f(0) = 0$, $f'(0) > 0$, the role of the Koebe function is played by the Pick functions $p_{\alpha}(z) := f_0^{-1}(\alpha f_0(z))$, $\alpha \in (0, 1)$, mapping \mathbb{D} onto $\mathbb{D} \setminus [-1, -r]$, $r = r(\alpha) \in (0, 1)$. In our case, it would be natural to expect that some functions of the form $f = h_1 \circ p_{\alpha} \circ h_2$, where $h_1, h_2 \in \operatorname{Aut}(\mathbb{D})$, are extremal.

Theorem 2. *For any $w_0 \in \partial\mathcal{V}(z_0, T) \setminus \{z_0\}$, there exists a unique $f = f_{w_0}$ satisfying conditions (i) – (iii) in Theorem 1 and such that $f_{w_0}(z_0) = w_0$. If $w_0 = \ell^{-1}(\zeta_0 + T)$, then f_{w_0} is a hyperbolic automorphism of \mathbb{D} , namely $f_{w_0}(z) = \ell^{-1}(\ell(z) + T)$. Otherwise, f_{w_0} is a conformal mapping of \mathbb{D} onto \mathbb{D} minus a slit along an analytic Jordan arc γ orthogonal to $\partial\mathbb{D}$, with $f'_{w_0}(1) = 1$. Moreover, $f_{w_0} = h_1 \circ p_{\alpha} \circ h_2$ for some $h_1, h_2 \in \operatorname{Aut}(\mathbb{D})$ and $\alpha \in (0, 1)$ if and only if $w_0 = \ell^{-1}(x_1^0 + \frac{T}{2} + i \arcsin a_{\pm}(T))$.*

Remark 1.1. Note that z_0 is a boundary point of the value region $\mathcal{V}(z_0, T)$, but does not belong to $\mathcal{V}(z_0, T)$. The proof of the above theorem, given in Section 4, shows that z_0 would be included, and this would be the only modification of the value region, if we replaced the equality $f'(-1) = e^T$ in condition (iii) of Theorem 1 by the *inequality* $f'(-1) \leq e^T$ and removed the requirement $f \neq \operatorname{id}_{\mathbb{D}}$ assuming as a convention that $\operatorname{id}_{\mathbb{D}}$ satisfies (ii). Note also that under the conditions of Theorem 1 modified in this way, $f(z_0) = z_0$ if and only if $f = \operatorname{id}_{\mathbb{D}}$, see Remark 2.3.

If $f \in \operatorname{Hol}(\mathbb{D}, \mathbb{D})$ has boundary regular fixed points at ± 1 , then replacing f by $h \circ f$, where h is a suitable hyperbolic automorphism with the same boundary fixed points, we may suppose that $\tau = 1$ is the Denjoy – Wolff point. In this way, as a corollary of Theorems 1 and 2 we easily deduce a sharp estimate for $f'(-1)f'(1)$, which was obtained earlier with the help of the extremal length method in [15, Section 4].

Corollary 1. *Let $z_0 \in \mathbb{D}$ and let $f \in \text{Hol}(\mathbb{D}, \mathbb{D})$ be a univalent function with boundary regular fixed points at 1 and -1 . Then*

$$(1.2) \quad \sqrt{f'(-1)f'(1)} \geq L(\sin \text{Im } \ell(z_0), \sin \text{Im } \ell(f(z_0))), \quad L(a, b) := \max \left\{ \frac{1+a}{1+b}, \frac{1-a}{1-b} \right\}.$$

Inequality (1.2) is sharp. The equality can occur only for hyperbolic automorphisms and functions of the form $f = h_1 \circ p_\alpha \circ h_2$, $h_1, h_2 \in \text{Aut}(\mathbb{D})$, $\alpha \in (0, 1)$.

Recently, the sharp value regions of $f \mapsto f(z_0)$ have been determined for other classes of univalent self-maps [22, 33, 35]. The main instrument is the classical parametric representation of univalent functions, going back to the seminal work by Loewner [27]. In this paper, we use a new variant of Loewner's parametric method, which is specific for functions satisfying conditions of Theorem 1. This variant of parametric representation was discovered quite recently, see [19, 20]. We discuss it in Section 3.

It is also worth mentioning that in [17], using another specific variant of the parametric representation, Goryainov obtained the sharp value region of $f \mapsto f'(0)$ in the class of all univalent $f \in \text{Hol}(\mathbb{D}, \mathbb{D})$, $f(0) = 0$, having a boundary regular fixed point at $\sigma = 1$ with a given value of $f'(1)$.

To complete the Introduction, we recall another related result announced by Goryainov [18]. Dropping the univalence requirement, one can study holomorphic self-maps $f : \mathbb{D} \rightarrow \mathbb{D}$ satisfying conditions (ii) and (iii) in Theorem 1 by using relationships between boundary regular fixed points and the Alexandrov–Clark measures. In particular, according to [18], the value region $\mathcal{D}(0, T)$ of $f \mapsto f(0)$ over all such self-maps f is the closed disk whose diameter is the segment $[0, \ell^{-1}(T)]$, with the boundary point $z_0 = 0$ excluded. Analyzing the functions delivering the boundary points of $\mathcal{D}(0, T)$, one can conclude that $\partial \mathcal{D}(0, T) \cap \partial \mathcal{V}(0, T) = \{0, \ell^{-1}(T)\}$.

2. HOLOMORPHIC SELF-MAPS OF THE UNIT DISK

In this section we cite some basic theory of holomorphic self-maps of \mathbb{D} . More details can be found, e.g., in the monograph [1].

Let $f \in \text{Hol}(\mathbb{D}, \mathbb{D})$ and $\sigma \in \partial \mathbb{D}$. According to the classical Julia–Wolff–Carathéodory Theorem, see, e.g., [1, Theorem 1.2.5, Proposition 1.2.6, Theorem 1.2.7], if

$$(2.1) \quad \alpha_f(\sigma) := \liminf_{\mathbb{D} \ni z \rightarrow \sigma} \frac{1 - |f(z)|}{1 - |z|} < +\infty,$$

then

$$(2.2) \quad \exists \angle \lim_{z \rightarrow \sigma} f(z) =: f(\sigma) \in \partial \mathbb{D}, \quad \exists \angle \lim_{z \rightarrow \sigma} \frac{f(z) - f(\sigma)}{z - \sigma} =: f'(\sigma) = \alpha_f(\sigma) \frac{f(\sigma)}{\sigma}, \quad \text{and}$$

$$(2.3) \quad \frac{|f(z) - f(\sigma)|^2}{1 - |f(z)|^2} \leq |f'(\sigma)| \frac{|z - \sigma|^2}{1 - |z|^2} \quad \text{for all } z \in \mathbb{D},$$

with the equality sign if and only if $f \in \text{Aut}(\mathbb{D})$. Note that in its turn, existence of the limits in (2.2) satisfying $f(\sigma) \in \partial \mathbb{D}$ and $f'(\sigma) \neq \infty$ immediately implies (2.1).

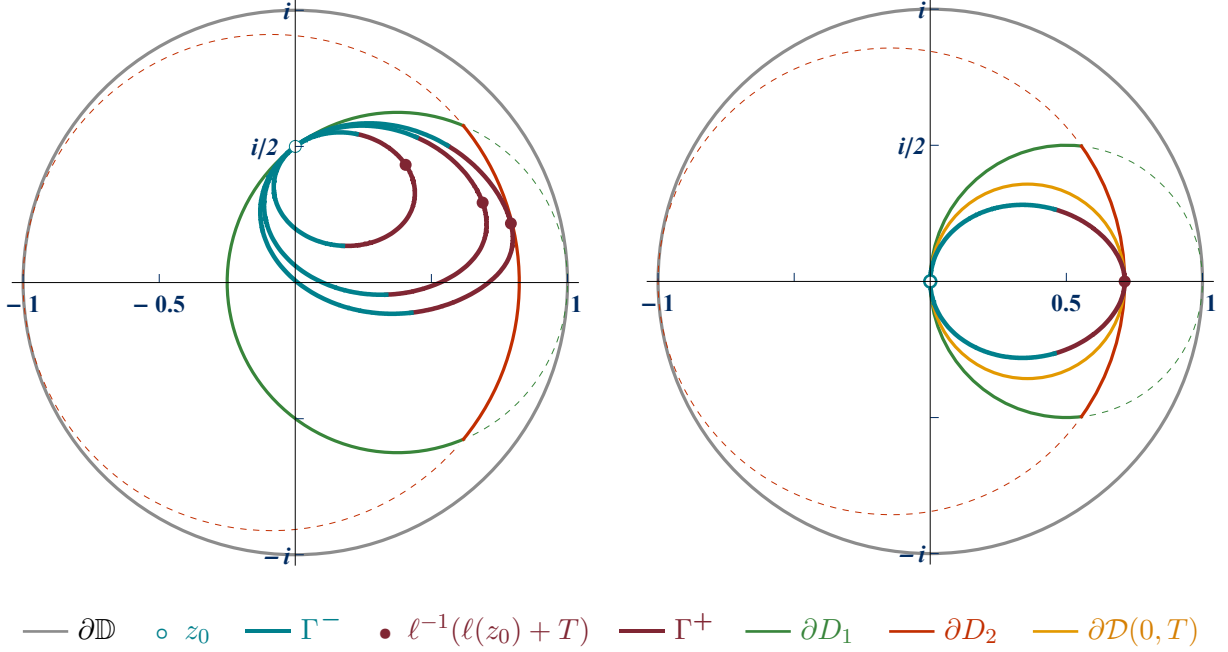


FIGURE 1. The value region $\mathcal{V}(z_0, T)$ and the disks D_1, D_2 for $z_0 := i/2$, $T \in \{\log 2, \log 4, \log 6\}$ and for $z_0 := 0$, $T := \log 6$. The right picture also shows the disk $\mathcal{D}(0, T)$. Notation Γ^\pm is explained in Section 4.

Definition 2.1. Points $\sigma \in \partial \mathbb{D}$ satisfying (2.2) are referred to as *regular contact points* of f . If in addition to (2.2), $f(\sigma) = \sigma$, then σ is said to be a *regular fixed point* of f . The number $f'(\sigma)$ is called the *angular derivative* of f at σ .

Among all fixed points (boundary and internal) of a self-map $f \neq \text{id}_{\mathbb{D}}$, there is one point of special importance for dynamics. On the one hand, if $f(\tau) = \tau$ for some $\tau \in \mathbb{D}$, then by the Schwarz Lemma, τ is the only fixed point of f in \mathbb{D} . If in addition, f is not an elliptic automorphism, then $|f'(\tau)| < 1$ and hence the sequence of iterates (f^{on}) , $f^{o1} := f$, $f^{o(n+1)} := f \circ f^{on}$, converges (to the constant function equal) to τ locally uniformly in \mathbb{D} . On the other hand, if f has no fixed points in \mathbb{D} , then by the Denjoy–Wolff Theorem, see, e.g. [1, Theorem 1.2.14, Corollary 1.2.16, Theorem 1.3.9], f has a unique boundary regular fixed point $\tau \in \partial \mathbb{D}$ such that $f'(\tau) \leq 1$ and moreover, $f^{on} \rightarrow \tau$ locally uniformly in \mathbb{D} as $n \rightarrow +\infty$.

Definition 2.2. The point τ above is referred to as the *Denjoy–Wolff point* of f .

Remark 2.3. Since the strict inequality holds in (2.3) unless $f \in \text{Aut}(\mathbb{D})$, a self-map f can have a fixed point in \mathbb{D} and a boundary regular fixed point σ with $f'(\sigma) \leq 1$ only if $f = \text{id}_{\mathbb{D}}$.

Remark 2.4. Let $f_n(z) := -\sigma H^{-1}(\alpha/H(z/\sigma) + \beta/(H(z/\sigma) + n))$, where $n \in \mathbb{N}$, $\alpha, \beta > 0$, and $H(z) := (1+z)/(1-z)$. Note that $f_n(\mathbb{D}) \subset \mathbb{D}$ for all $n \in \mathbb{N}$ and that $f_n(z) \rightarrow$

$f(z) := (z + c)/(1 + \bar{c}z)$, $c := \sigma(1 - \alpha)/(1 + \alpha)$, locally uniformly in \mathbb{D} as $n \rightarrow +\infty$. Moreover, $f_n(\sigma) = f(\sigma) = \sigma$ and $f'_n(\sigma) = \alpha + \beta$ for all $n \in \mathbb{N}$, but $f'(\sigma) = \alpha$. This example shows that the map $f \mapsto f'(\sigma)$ is not continuous. However, it turns out to be semicontinuous in the following sense. Suppose that $f_n(z) \rightarrow f(z)$ as $n \rightarrow +\infty$ and that $\sigma \in \partial\mathbb{D}$ is a boundary regular fixed point of $f_n \in \text{Hol}(\mathbb{D}, \mathbb{D})$ for all $n \in \mathbb{N}$ with $\alpha := \liminf_{n \rightarrow +\infty} f'_n(\sigma) < +\infty$. Then passing in Julia's inequality (2.3) applied for functions f_n to the limit, we conclude that f satisfies (2.3) with $|f'(\sigma)|$ replaced by α . It follows that $\alpha_f(\sigma) \leq \alpha < +\infty$. Therefore, either $f \equiv \sigma$ or $f \in \text{Hol}(\mathbb{D}, \mathbb{D})$ and σ is a regular boundary fixed point of f with $f'(\sigma) \leq \alpha$. As a consequence, the set of all $f \in \text{Hol}(\mathbb{D}, \mathbb{D})$ sharing two different boundary regular fixed points σ_1 and σ_2 and satisfying $f'(\sigma_j) \leq \alpha_j < +\infty$, $j = 1, 2$, is compact.

According to inequality (2.3), the value region $\mathcal{V}(z_0, T)$ in Theorem 1 lies in the intersection of two closed disks $D_1, D_2 \subset \bar{\mathbb{D}}$ whose boundaries pass through z_0 and $\tau = 1$ and through $\ell^{-1}(\ell(z_0) + T)$ and $\sigma = -1$, respectively. Comparison of $\mathcal{V}(z_0, T)$ with $D_1 \cap D_2$ is shown in Figure 1. On the right picture, for which $z_0 = 0$, we also place the value range $\mathcal{D}(0, T)$ of $f \mapsto f(0)$ over all holomorphic but not necessary injective maps $f : \mathbb{D} \rightarrow \mathbb{D}$ satisfying conditions (ii) and (iii) in Theorem 1, see [18].

3. PARAMETRIC REPRESENTATION

Denote the class of all $f \in \text{Hol}(\mathbb{D}, \mathbb{D})$ satisfying conditions (i)–(iii) in Theorem 1 by $\mathfrak{U}(T)$. The following theorem, proved in [20], gives a parametric representation for $\mathfrak{U}(T)$ in terms of a Loewner–Kufarev-type ODE.

Theorem 3 ([20, Corollary 1.2]). *The class $\mathfrak{U}(T)$ coincides with the set of all functions representable in the form $f(z) = w_z(T)$ for all $z \in \mathbb{D}$, where $w_z(t)$ is the unique solution to the initial value problem*

$$(3.1) \quad \frac{dw_z}{dt} = \frac{1}{4}(1 - w_z)^2(1 + w_z)q(w_z, t), \quad t \in [0, T], \quad w_z(0) = z,$$

with some function $q : \mathbb{D} \times [0, T] \rightarrow \mathbb{C}$ satisfying the following conditions:

- (i) for every $z \in \mathbb{D}$, $q(z, \cdot)$ is measurable on $[0, T]$;
- (ii) for a.e. $t \in [0, T]$, $q(\cdot, t)$ has the following integral representation

$$(3.2) \quad q(z, t) = \int_{\partial\mathbb{D} \setminus \{1\}} \frac{1 - \kappa}{1 + \kappa z} d\nu_t(\kappa),$$

where ν_t is a probability measure on $\partial\mathbb{D} \setminus \{1\}$.

Remark 3.1. A related parametric representation for a class of univalent self-maps of a strip was considered in [13].

Remark 3.2. In many cases, it is more convenient to deal with the union $\mathfrak{U}'(T) := \bigcup_{0 \leq T' \leq T} \mathfrak{U}(T')$, where we define $\mathfrak{U}(0) := \{\text{id}_{\mathbb{D}}\}$. Indeed, it is evident from the argument of Remark 2.4 that in contrast to $\mathfrak{U}(T)$, the class $\mathfrak{U}'(T)$ is compact. Moreover, it is easy

to see that Theorem 3 gives representation of $\mathfrak{U}'(T)$ if all probability measures ν_t in (3.2) are replaced with all positive Borel measures ν_t satisfying

$$(3.3) \quad \nu_t(\partial\mathbb{D} \setminus \{1\}) \in [0, 1].$$

Note that the possibility of $\nu_t = 0$ is not excluded.

Remark 3.3. Obviously, the right-hand side of (3.1) can be written as $G(w_z, t)$, where $G(z, t) := \frac{1}{4}(1 - z)^2(1 + z)q(z, t)$ with q satisfying conditions (i) and (ii) in Theorem 3. By [19, Theorem 1], $G(\cdot, t)$ is an infinitesimal generator in \mathbb{D} for each $t \in [0, T]$. For simplicity, extend G to all $t \geq 0$ by setting $G(z, t) \equiv 0$ for any $t > T$. Then according to the general theory of Loewner – Kufarev-type equations, see [4, Sections 3–5], for any $s \geq 0$ and any $z \in \mathbb{D}$, the initial value problem $dw/dt = G(w, t)$, $t \geq s$, $w(s) = z$, has a unique solution $w = w_{z,s}(t)$ defined for all $t \geq s$ and the functions $\varphi_{s,t}(z) := w_{z,s}(t)$, $z \in \mathbb{D}$, $t \geq s \geq 0$, form an evolution family, see [4, Definition 3.1].

Proposition 1. *Let $\vartheta : [0, T] \rightarrow (-\pi, \pi) \setminus \{0\}$, $T > 0$, be a C^1 -smooth function. Suppose that in the conditions of Theorem 3, $d\nu_t(e^{i\theta}) = \delta(\theta - \vartheta(t)) d\theta$ for all $t \in [0, T]$, where δ stands for the Dirac delta function. Then f maps \mathbb{D} onto $\mathbb{D} \setminus \gamma$, where γ is a slit in \mathbb{D} , i.e. γ is the image of a homeomorphism $\gamma : [0, 1] \mapsto \overline{\mathbb{D}}$ with $\gamma([0, 1)) \subset \mathbb{D}$ and $\gamma(1) \in \partial\mathbb{D}$.*

Moreover:

- (i) *if ϑ is a real-analytic function on $[0, T]$, then γ is a real-analytic Jordan arc orthogonal to \mathbb{D} ;*
- (ii) *γ is a circular arc or a straight line segment orthogonal to $\partial\mathbb{D}$ if and only if*

$$(3.4) \quad \lambda(t) := i \frac{1 + e^{i\vartheta(t)}}{1 - e^{i\vartheta(t)}} = C_1 e^{-t/2} \left(C_2 e^{t/2} + \sqrt{C_2^2 (e^t - 1) + 1} \right)^3$$

for all $t \in [0, T]$ and some constants $C_1, C_2 \in \mathbb{R}$, $C_1 \neq 0$.

Proof. In the conditions of the proposition, (3.1) takes the following form:

$$(3.5) \quad \frac{dw_z}{dt} = \frac{1}{4}(1 - w_z)^2(1 + w_z) \frac{1 - e^{i\vartheta(t)}}{1 + e^{i\vartheta(t)}w_z}, \quad t \in [0, T], \quad w_z(0) = z.$$

The change of variables $\omega_z := H(w_z)$, where $H(w) := i(1+w)/(1-w)$ maps \mathbb{D} conformally onto $\mathbb{H} := \{\omega : \operatorname{Im} \omega > 0\}$, transforms the above problem to

$$(3.6) \quad \frac{d\omega_z}{dt} = \frac{\omega_z}{1 - \lambda(t)\omega_z}, \quad t \in [0, T], \quad \omega_z(0) = H(z),$$

where $\lambda(t) := H(e^{i\vartheta(t)})$ for all $t \in [0, T]$. Making further change of variables

$$\hat{\omega}_z(t) := \omega_z(t) + \int_0^t \frac{ds}{\lambda(s)}, \quad \xi(t) := \frac{1}{\lambda(t)} + \int_0^t \frac{ds}{\lambda(s)}, \quad \tau = v(t) := \frac{1}{2} \int_0^t \frac{ds}{\lambda(s)^2},$$

we obtain the chordal Loewner equation

$$(3.7) \quad \frac{d\hat{\omega}_z}{d\tau} = \frac{2}{\xi - \hat{\omega}_z}, \quad \tau \in [0, v(T)], \quad \hat{\omega}_z(0) = H(z).$$

The geometry of solutions to (3.7) is well-studied, see, e.g., [25, 28, 34, 21, 37]; see also [23]. In particular, since the function $s \mapsto \xi(v^{-1}(s))$ is C^1 -smooth, it follows that $z \mapsto \hat{w}_z(T)$ maps \mathbb{D} onto \mathbb{H} minus a slit along some Jordan arc γ_0 . Taking into account that $w_z(T) = H^{-1}(\hat{w}_z(T) - C)$, where $C := \int_0^T \lambda(t)^{-1} dt$, this proves the first part of the proposition.

If ϑ is real-analytic, then $s \mapsto \xi(v^{-1}(s))$ is real-analytic on $[0, T]$ as well, and hence by [26, Theorem 1.4], γ_0 is a real-analytic Jordan arc. Moreover, the argument of [26, Section 6.1] shows that in such a case, γ_0 is orthogonal to \mathbb{R} . This proves (i).

It remains to prove (ii). Suppose that γ is a circular arc or a straight line segment orthogonal to $\partial\mathbb{D}$. Then we can find a linear-fractional transformation H_* of \mathbb{D} onto \mathbb{H} such that $H_*(\gamma) = [0, i]$. Let $(\varphi_{s,t})$ be the evolution family associated with equation (3.5), see Remark 3.3. Note that $\varphi_{t,T}(\mathbb{D}) \supset \varphi_{t,T}(\varphi_{0,t}(\mathbb{D})) = \varphi_{0,T}(\mathbb{D}) = f(\mathbb{D})$ for any $t \in [0, T]$. It follows that the intersection of a sufficiently small neighbourhood of $H_*^{-1}(\infty)$ with $\partial\varphi_{t,T}(\mathbb{D})$ is an open arc of $\partial\mathbb{D}$ containing $H_*^{-1}(\infty)$. Therefore, for each $t \in [0, T]$, there exists a unique $h_t \in \text{Aut}(\mathbb{D})$ such that $g_t := H_* \circ \varphi_{t,T} \circ h_t \circ H_*^{-1} \in \text{Hol}(\mathbb{H}, \mathbb{H})$ satisfies the Laurent expansion $g_t(z) = z - c(t)/z + \dots$ at ∞ with some $c(t) \in \mathbb{R}$.

Denote $H_t := H_* \circ h_t^{-1}$ for all $t \in [0, T]$. By construction, $\mathbb{H} \setminus [0, i] = g_0(\mathbb{H}) \subset g_t(\mathbb{H}) \subset g_T(\mathbb{H}) = \mathbb{H}$ for all $t \in [0, T]$. Thanks to continuity of ϑ , the function $t \mapsto c(t)$ is C^1 -smooth. Therefore, according to the classical result [24] by Kufarev et al, see also [12], for any $z \in \mathbb{D}$, $\tilde{w}_z(t) := g_t^{-1} \circ g_0(H_0(z))$, $t \in [0, T]$, is the unique solution to the initial value problem $d\tilde{w}_z/dt = -c'(t)/\tilde{w}_z$, $\tilde{w}_z(0) = H_0(z) \in \mathbb{H}$.

By construction, $\tilde{w}_z(t) = H_t(w_z)$ for all $t \in [0, T]$ and all $z \in \mathbb{D}$. Comparing the differential equations for \tilde{w}_z and w_z , one can conclude that for all $t \in [0, T]$,

$$(3.8) \quad H_t(w) := \frac{\lambda(t)H(w) - 1}{a(t)(\lambda(t)H(w) - 1) + b(t)}$$

with real coefficients $a(t)$ and $b(t)$ satisfying

$$(3.9) \quad da/dt = a^3/b^2, \quad db/dt = -3a + b + 3a^2/b, \quad t \in [0, T],$$

and such that $\lambda'(t)/\lambda(t) = 1 - 3a(t)/b(t)$ and $b(t)\lambda(t) > 0$ for all $t \in [0, T]$. System (3.9) can be solved by introducing a new unknown function $k(t) := a(t)/b(t)$. In this way, one can easily check that λ must be of the form (3.4).

Conversely, if λ is given by (3.4), then system (3.9) has a real-valued solution satisfying $\lambda'(t)/\lambda(t) = 1 - 3a(t)/b(t)$ and $b(t)\lambda(t) > 0$ for all $t \in [0, T]$. It follows that for any $z \in \mathbb{D}$, the function $\tilde{w}_z(t) := H_t(w_z(t))$, where H_t is given by (3.8), is a solution to $d\tilde{w}_z/dt = -1/(b(t)^2 \tilde{w}_z)$, $t \in [0, T]$, $\tilde{w}_z(0) = H_0(z) \in \mathbb{H}$. Solving the latter initial value problem for \tilde{w}_z , we conclude that the image of the map $\mathbb{D} \ni z \mapsto \tilde{w}_z(T)$ is the domain $\mathbb{H} \setminus [0, i\sqrt{Q_T}]$, $Q_T := 2 \int_0^T b(t)^{-2} dt$. Thus, $\gamma = H_T^{-1}([0, i\sqrt{Q_T}])$ is a circular arc or a straight line segment orthogonal to $\partial\mathbb{D}$. The proof is now complete. \square

4. PROOF OF THE MAIN RESULTS

In this section we prove Theorems 1 and 2. Fix $T > 0$. We start by considering the problem to determine the compact value region $\{f(z_0) : f \in \mathfrak{U}(T)\}$. Thanks to

Theorem 3 and Remark 3.2, it coincides with the reachable set $\{w_{z_0}(T)\}$ of the controllable system (3.1) in which the measure-valued control $t \mapsto \nu_t$ satisfies (3.3). The change of variables

$$\zeta = \ell(w), \quad \lambda = i \frac{1 + \kappa}{1 - \kappa},$$

reduces our problem to finding the reachable set $\Omega'_T := \{\zeta(T)\}$ for the following controllable system

$$(4.1) \quad \frac{d\zeta}{dt} = \int_{\mathbb{R}} \frac{d\mu_t(\lambda)}{1 - i\lambda e^\zeta}, \quad t \in [0, T]; \quad \zeta|_{t=0} = \zeta_0 := \ell(z_0),$$

where μ_t 's are positive Borel measures on \mathbb{R} with $\mu_t(\mathbb{R}) \leq 1$. By using the prime in the notation Ω'_T we emphasize that this reachable set corresponds to the class $\mathfrak{U}'(T)$.

Denote $x_1 := \operatorname{Re} \zeta$ and $x_2 := \operatorname{Im} \zeta$. Note that $x_2 \in (-\frac{\pi}{2}, \frac{\pi}{2})$. For any fixed $\zeta = x_1 + ix_2 \in \mathbb{S}$, the range of the right-hand side in (4.1), regarded as a function of the measure μ_t , is the disk

$$\left\{ \omega \in \mathbb{C} : \left| \omega - \frac{e^{-ix_2}}{2 \cos x_2} \right| \leq \frac{1}{2 \cos x_2} \right\}.$$

Therefore, replacing the measure-valued control $t \mapsto \mu_t$ with the complex-valued control

$$u(t) := 2e^{ix_2} \cos x_2 \int_{\mathbb{R}} \frac{d\mu_t(\lambda)}{1 - i\lambda e^{x_1 + ix_2}},$$

we can rewrite (4.1) in the following form

$$(4.2) \quad \frac{dx_1}{dt} = \operatorname{Re} \frac{u(t)e^{-ix_2}}{2 \cos x_2} = \frac{1}{2} \operatorname{Re} u(t) + \frac{\operatorname{tg} x_2}{2} \operatorname{Im} u(t), \quad x_1(0) = x_1^0 := \operatorname{Re} \zeta_0,$$

$$(4.3) \quad \frac{dx_2}{dt} = \operatorname{Im} \frac{u(t)e^{-ix_2}}{2 \cos x_2} = \frac{1}{2} \operatorname{Im} u(t) - \frac{\operatorname{tg} x_2}{2} \operatorname{Re} u(t), \quad x_2(0) = x_2^0 := \operatorname{Re} \zeta_0,$$

where $u : [0, T] \rightarrow U := \{u : |u - 1| \leq 1\}$ is an arbitrary measurable function.

Introduce the Hamilton function

$$\mathcal{H}(x_1, x_2, \Psi_1, \Psi_2, u) := \Psi_1 \operatorname{Re} \frac{ue^{-ix_2}}{2 \cos x_2} + \Psi_2 \operatorname{Im} \frac{ue^{-ix_2}}{2 \cos x_2} = \operatorname{Re} \frac{ue^{-ix_2}(\Psi_1 - i\Psi_2)}{2 \cos x_2},$$

where Ψ_1, Ψ_2 satisfy the adjoint system of ODEs

$$(4.4) \quad \frac{d\Psi_1}{dt} = -\frac{\partial \mathcal{H}}{\partial x_1} = 0, \quad \frac{d\Psi_2}{dt} = -\frac{\partial \mathcal{H}}{\partial x_2} = -\operatorname{Im} \frac{u(t)(\Psi_1 - i\Psi_2)}{2 \cos^2 x_2}.$$

Boundary points of the reachable set Ω'_T , forming a dense subset of $\partial\Omega'_T$, are generated by the driving functions u^* satisfying the necessary optimal condition in the form of Pontryagin's maximum principle,

$$(4.5) \quad \max_{u \in U} \mathcal{H}(x_1(t), x_2(t), \Psi_1(t), \Psi_2(t), u) = \mathcal{H}(x_1(t), x_2(t), \Psi_1(t), \Psi_2(t), u^*(t))$$

for all $t \in [0, T]$, see, e.g., [32]. Trajectories $(x_1(t), x_2(t))$ in (4.5) are optimal in the reachable set problem, and $(\Psi_1(t), \Psi_2(t))$ satisfy the adjoint system (4.4) with the optimal

trajectories. In particular, $(\Psi_1(t), \Psi_2(t))$ does not vanish, and hence the maximum in (4.5) is attained at the unique point $u^* = 1 + e^{i(x_2 + \varphi)}$, where $\varphi := \arg(\Psi_1 + i\Psi_2)$. Therefore, from (4.2)–(4.4) for the optimal trajectories we obtain

$$(4.6) \quad \frac{dx_1}{dt} = \frac{\cos \varphi + \cos x_2}{2 \cos x_2}, \quad x_1(0) = x_1^0,$$

$$(4.7) \quad \frac{dx_2}{dt} = \frac{\sin \varphi - \sin x_2}{2 \cos x_2}, \quad x_2(0) = x_2^0,$$

$$(4.8) \quad \frac{d\Psi_1}{dt} = 0,$$

$$(4.9) \quad \frac{d\Psi_2}{dt} = \frac{\sin \varphi - \sin x_2}{2 \cos^2 x_2} |\Psi_1 - i\Psi_2|.$$

System (4.6)–(4.9) is invariant w.r.t. multiplication of (Ψ_1, Ψ_2) by a positive constant. Therefore, we may assume that either $\Psi_1 \equiv 0$, or $\Psi_1 \equiv 1$, or $\Psi_1 \equiv -1$.

If $\Psi_1 \equiv 0$, then $\varphi = \pm\pi/2$ and we easily get that for all $t \geq 0$,

$$(4.10) \quad x_1(t) = x_1(0) + t/2, \quad \sin x_2(t) = a_{\pm}(t) := e^{-t/2} \sin x_2(0) \pm (1 - e^{-t/2}).$$

Now let $\Psi_1 \equiv 1$. Then $\varphi \in (-\pi/2, \pi/2)$ and equation (4.9) takes the following form

$$(4.11) \quad \frac{d\varphi}{dt} = \frac{\sin \varphi - \sin x_2}{2 \cos^2 x_2} \cos \varphi = \frac{\cos \varphi}{\cos x_2} \frac{dx_2}{dt}.$$

System (4.6), (4.11) admits the first integral

$$I(x_2, \varphi) := \frac{1 - \sin \varphi}{1 + \sin \varphi} \frac{1 + \sin x_2}{1 - \sin x_2} > 0,$$

and as a result it can be integrated in quadratures. Namely, if $C := I(x_2(0), \varphi(0)) \neq 1$, we obtain the following identities

$$(4.12) \quad B_1(t) - CB_2(t) = (C - 1)t/2,$$

$$(4.13) \quad x_1(t) - x_1(0) = \frac{B_1(t) - \sqrt{C}B_2(t)}{\sqrt{C} - 1},$$

$$\text{where } B_1(t) := \log \frac{1 - \sin x_2(t)}{1 - \sin x_2(0)}, \quad B_2(t) := \log \frac{1 + \sin x_2(t)}{1 + \sin x_2(0)}.$$

Excluding C from (4.12), (4.13) and setting $t := T$ gives

$$(4.14) \quad x_1(T) = x_1(0) + \frac{1}{2} \left(T + \sqrt{(T + 2B_1(T))(T + 2B_2(T))} \right) \\ = x_1(0) + \frac{T}{2} + \sqrt{R(\sin x_2(T), T)},$$

where we took into account that according to (4.12),

$$\frac{d}{dt} (t + 2B_1(t)) = \frac{2C}{1 + \sin x_2(t) + C(1 - \sin x_2(t))} > 0$$

and therefore, $T + 2B_1(T) > 0$.

For $C = 1$, we have $\varphi(t) = x_2(t)$ and hence $d\varphi/dt = dx_2/dt = 0$, $dx_1/dt = 1$. Therefore, if $C = 1$, then (4.12) and (4.14) hold as well.

Since $C > 0$, from (4.12) we obtain that $x_2(T) \in J(T) := (\arcsin a_-(T), \arcsin a_+(T))$. On the other hand, for any $x \in J(T)$ there exists a unique $C = C(x) > 0$ that verifies (4.12) with T and x substituted for t and $x_2(t)$, respectively. Solving $I(x_2(0), \varphi(0)) = C(x)$ provides us with the initial condition in equation (4.11) for which $x_2(T) = x$.

Investigating the case $\Psi_1 \equiv -1$ in a similar way, we conclude that $\partial\Omega'_T$ is the union of the two Jordan arcs

$$\Gamma^\pm(T) := \left\{ x_1 + ix_2 \in \mathbb{S} : a_-(T) \leq \sin x_2 \leq a_+(T), \ x_1 = x_1^0 + \frac{T}{2} \pm \sqrt{R(\sin x_2, T)} \right\},$$

which do not intersect except for the common end-points $\omega^\pm := x_1^0 + T/2 + i \arcsin a_\pm(T)$, delivered by solutions (4.10). Taking into account that by the very definition, $\mathfrak{U}'(T') \subset \mathfrak{U}'(T)$ for any $T' \in [0, T]$, it follows that $\Omega'_T = V(\zeta_0, T)$.

The next step in the proof is to pass from the class $\mathfrak{U}'(T)$ to the class $\mathfrak{U}(T)$. In the problem of finding the value region of the functional $f \mapsto f(z_0)$, this is equivalent to replacing the range U of the admissible controls u in (4.2)–(4.3) by $U \setminus \{0\}$. Denote by Ω_T the corresponding reachable set. By re-scaling the time, the problem to find $\Omega_{T'}$, $T' \in (0, T)$, can be restated as the reachable set problem at the same time T and for the same controllable system, but with the value range of admissible controls restricted to $\alpha(U \setminus \{0\})$, $\alpha := T'/T$. Note also that $\Gamma^+(T) \cup \Gamma^-(T) \setminus \{\zeta_0\} \subset \Omega_T$ for any $T > 0$. Since $\alpha(U \setminus \{0\}) \subset U \setminus \{0\}$ for any $\alpha \in (0, 1)$, it follows that

$$\Gamma^+(T') \cup \Gamma^-(T') \setminus \{\zeta_0\} \subset \Omega_{T'} \subset \Omega_T \quad \text{for any } T \in (0, T].$$

Thus $\Omega_T = V(\zeta_0, T) \setminus \{\zeta_0\}$, which completes the proof of Theorem 1.

To prove Theorem 2, we have to identify the functions delivering the boundary points of $\mathcal{V}(z_0, T)$. They correspond to the controls u^* satisfying Pontryagin's maximum principle (4.5). It is easy to see from the above argument that every point $\omega \in \partial\Omega'_T \setminus \{\zeta_0\}$ corresponds to a unique control, which is C^1 -smooth and takes values on $\partial U \setminus \{0\}$. It follows that the corresponding measures μ_t in (4.1) and the measures ν_t in the Loewner-type representation (3.1), (3.2) are also unique. They are probability measures concentrated at one point that moves smoothly with t . Namely, $d\mu_t(\lambda) = \delta(\lambda - \lambda^*(t)) d\lambda$, where

$$(4.15) \quad \lambda^*(t) := \frac{1 - 2 \cos x_2(t) / (e^{-ix_2(t)} + e^{i\varphi(t)})}{ie^{x_1(t) + ix_2(t)}} = e^{-x_1(t)} \frac{\sin \frac{\varphi(t) - x_2(t)}{2}}{\cos \frac{\varphi(t) + x_2(t)}{2}}.$$

The point $\omega = \omega_0 := \zeta_0 + T \in \Gamma^+$ corresponds to $C = 1$, in which case $\varphi(t) = x_2(t)$ for all $t \in [0, T]$ and hence $\lambda^*(t) \equiv 0$. Therefore, from (4.1) we see that the unique $f \in \mathfrak{U}(T)$ delivering the boundary point $\ell^{-1}(\omega_0)$ of $\mathcal{V}(z_0, T)$ is the hyperbolic automorphism

$$f(z) = \frac{z + c(T)}{1 + c(T)z}, \quad c(T) := \frac{e^T - 1}{e^T + 1}, \quad \text{for all } z \in \mathbb{D}.$$

For the common end-points ω^\pm of Γ^+ and Γ^- , which correspond to $\varphi = \pm\pi/2$, formula (4.15) simplifies to $\lambda^*(t) = \pm e^{-x_1(t)}$. In view of (4.10), the latter expression coincides with $\lambda(t)$ given by (3.4) if we set $C_1 := \pm e^{-x_1^0}$ and $C_2 := 0$. Taking into account the correspondence between μ_t and ν_t and applying Proposition 1, we conclude that the unique functions $f \in \mathfrak{U}(T)$ delivering the points $\ell^{-1}(\omega^\pm)$ map \mathbb{D} onto \mathbb{D} minus a slit along a circular arc or a segment of a straight line orthogonal to $\partial\mathbb{D}$.

It remains to compare $\lambda^*(t)$ given by (4.15) with $\lambda(t)$ given by (3.4) for the case $\omega \in \partial\Omega_T \setminus \{\zeta_0, \omega_0, \omega^+, \omega^-\}$. Suppose $\omega \in \Gamma^+ \setminus \{\omega_0, \omega^+, \omega^-\}$. Using equations (4.6), (4.7), (4.11) and taking into account the first integral $I(x_2, \varphi) = C$, we find that

$$\left(1 + 2 \frac{d}{dt} \log \lambda^*(t)\right)^2 = \left(\frac{\cos \varphi(t)}{\cos x_2(t)}\right)^2 = \frac{C(1 - a^2)}{((1 + C)a + (1 - C)a^2)^2}, \quad a := \sin x_2(t),$$

while $\left(1 + 2(d/dt) \log \lambda(t)\right)^2 = 9C_2^2 e^t / (1 + C_2^2(e^t - 1))$. However, according to (4.12), e^t cannot be expressed as a rational function of $\sin x_2(t)$. This shows that λ^* is not of the form (3.4) and hence, by Proposition 1, the unique function $f \in \mathfrak{U}(T)$ that delivers the boundary point $\ell^{-1}(\omega)$ maps \mathbb{D} onto \mathbb{D} minus a slit along a real-analytic arc γ orthogonal to $\partial\mathbb{D}$ but different from a circular arc or a segment of a straight line. A similar argument applied to the case $\omega \in \Gamma^- \setminus \{\zeta_0, \omega^+, \omega^-\}$ completes the proof of Theorem 2. \square

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